Draft

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1 Introduction

Definition 1 Epigraph
We let $u : R^d \to R$ is a continuous function, the epigraph of u is a subspace of R^{d+1}

$$epi(u) := \{(x, \xi) \in R^d \times R | \xi \geq u(x)\}$$

Definition 2 Legendre-Fenchel transformation
Given an open set $\Omega \subset R^d$ and function $u : \Omega \to R$, the Legendre-Fenchel transformation of u is $u^* : R^d \to R \cup \{+\infty\}$ which defined below

$$u^*(p) := \sup_{x \in \Omega} \langle p, x \rangle - u(x)$$

Here p is an element of R^d and $\langle . \rangle$ can be view as inner product. u^* is a function of p with respect to u. We can define Legendre dual through this transformation, thus the p become an element of $(R^d)^*$ which is the dual space of R^d, also called linear functional. Therefore, $\langle p, x \rangle$ is equal to p(x), because the inner product is equivalent to linear functional in finite dimension space according to Rize representation theory. $u^* : (R^d)^* \to R$
We set f is a functional of $(R^d)^*$, it is characterized by p, and $x \in R^d$

$$f(x) = p_1 x_1 + p_2 x_2 + ... + p_d x_d = \langle p, x \rangle$$